CLASSIFICATION OF ROTATIONS ON THE TORUS $\mathbb{T}^2$

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Abstract. We consider a rotation on the torus $\mathbb{T}^2$. We classify these rotations along their complexity functions. This can be seen as a generalization of Morse Hedlund theorem to the dimension two.

1. Introduction

The rotations of the torus $\mathbb{T}^1$ have been extensively studied since the work of Morse, Hedlund [11]. The associated dynamical system is an exchange of two intervals. If we code the intervals by two letters, the orbit of a point becomes an infinite word. This word can have two shapes. It can be a periodic word, or it is a sturmian word. Its complexity is thus either constant or equal to $n+1$. Here we consider the similar problem for the two dimensional case. This map can be seen as a billiard map, in a fixed direction, inside the cube. The computation of the complexity has been made if the direction satisfy some algebraic conditions. Under these assumptions the complexity equals n^2+n+1. The first proof was given in [1, 2], and a general proof in dimension $s \geq 3$ appears in [3]. Moreover we give another proof of the 3 dimensional result in [4], and we remark that the proof of [1, 2] is false: there exists minimal direction with a complexity less than n^2+n+1. In this paper we give the complete classification of the complexity of a rotation.

1.1. Statement of the theorem.

Theorem 1. *We fix an orthonormal basis of* $\mathbb{R}^3$ *such that the edges of the cube are parallel to the coordinate axis. Let* $\omega = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}$ *be a unit vector of* $\mathbb{R}^3$ *such that* $w_i \neq 0$ *for all* i. *Denote* $\alpha = \frac{\omega_2}{\omega_1}, \beta = \frac{\omega_3}{\omega_1}$. *Then the directional complexity can be computed, and we obtain:*

(1)• *If* α, β *are rational numbers, then there exists* $C > 0, n_0$ *such that* $p(n,\omega) = C$ *for all integer* $n \geq n_0$.

(2)• *If* α *is an irrational number, and* β *is a rational number, then there exists* C *such that* $p(n,\omega) \leq Cn$.

(3)• *If* α, β *are irrational numbers such that* $1, \alpha, \beta$ *are linearly dependent over* $\mathbb{Q}$, *then there exists* C *such that* $p(n,\omega) \leq Cn$ *for all* n.

(4)• *If* $\alpha, \beta, 1$ *are linearly independent over* $\mathbb{Q}$, *and if* $\alpha^{-1}, \beta^{-1}, 1$ *are linearly dependant over* $\mathbb{Q}$, *then there exists* $C \in]0;1[$ *such that* $p(n,\omega) \sim Cn^2$.

(5)• *In the last case we obtain* $p(n,\omega) = n^2+n+1$.

Corollary 2. • *In case* (2) *two directions with the same value of* β *have the same complexity.*

• *For the third case, two directions in the same plane have the same complexity. It means if two directions* ω, θ *satisfy* $a\omega_1 + b\omega_2 + c\omega_3 = a\theta_1 + b\theta_2 + c\theta_3 = 0$ *with* $a, b, c \in \mathbb{Z}$*, then* $p(n, \omega) = p(n, \theta)$*.*

• *In case* (4)*, we can compute the constant* C*. If* (ω_i) *follow the equation* $\frac{A}{\omega_1} = \frac{B}{\omega_2} + \frac{C}{\omega_3}$*, with* $A, B, C \in \mathbb{N}$ *then we obtain*

$$C = 1 - \frac{1}{A(\alpha + \beta + 1)}.$$

The other cases are obtained by permutation.

Remark 3. *The cases* (2) *and* (3) *correspond to the same algebraic condition. We separate the cases, since in the first, an orthonormal projection on a face give a periodic word.*

The definitions are given in the following section.

Outline of the paper: In Section 2 and 3 we recall some usual facts of billiard and word combinatorics. In Section 4 we begin the proof of the theorem.

2. Background

2.1. **Billiard.** We recall some fact of billiard theory. For this subsection we refer to [12] or [10]. Consider a cube C.

• First we define the billiard map: A billiard ball, i.e. a point mass, moves inside C with unit speed along a straight line until it reaches the boundary ∂C, then instantaneously changes direction according to the mirror law, and continues along the new line. Thus the billiard map T is defined on a subset X of $\partial C \times \mathbb{PR}^2$:

$$T : X \to \partial C \times \mathbb{PR}^2.$$

The following tool is very useful for the billiard, it is called the unfolding. Consider a billiard trajectory in a polyhedron. To draw the orbit, we must reflect the line each time it hits a face of the polyhedron. The unfolding consists to reflect the polyhedron through the face and continue on the same line.

Remark 4. *In the following, if we use the term direction, we will consider an unit vector of* $\mathbb{R}^3$*.*

In this method the billiard orbit of (m, ω) is viewed as the sequence of intersections of the line $m + \mathbb{R}\omega$ with the lattice $\mathbb{Z}^3$, see figure.

By unfolding process the study of billiard orbit can be made by the study of the translation on the torus $\mathbb{R}^3/\mathbb{Z}^3$.

Definition 5. *In the following, we denote by* T_ω *the translation of vector* ω *in the torus* $\mathbb{R}^3/\mathbb{Z}^3$*.*

• Symbolic dynamic.
Label the faces of C by three symbols from a finite alphabet $\mathcal{A}$ such that two parallel faces of the cube are coded by the same symbols. To the orbit of a point in a direction ω, we associate a word in the alphabet $\mathcal{A}$ defined by the sequence of faces of the billiard trajectory.

Definition 6. *The set of points (m, ω) such that for all integer n $T^n(m, \omega) \in X$ is denoted by X_∞. The infinite word associate to a point (m, ω) in X_∞ is denoted by $v_{m,\omega}$.*

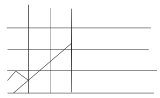

FIGURE 1. Unfolding

2.2. **Combinatorics.**

Definition 7. *Let $\mathcal{A}$ be a finite set called the alphabet. By a language L over $\mathcal{A}$ we mean always a factorial extendable language: a language is a collection of sets $(L_n)_{n\geq 0}$ where the only element of L_0 is the empty word, and each L_n consists of words of the form $a_1a_2\dots a_n$ where $a_i \in \mathcal{A}$ and such that for each $v \in L_n$ there exist $a, b \in \mathcal{A}$ with $av, vb \in L_{n+1}$, and for all $v \in L_{n+1}$ if $v = au = u'b$ with $a, b \in \mathcal{A}$ then $u, u' \in L_n$.*
The complexity function $p : \mathbb{N} \to \mathbb{N}$ is defined by $p(n) = card(L_n)$.

Let $\mathcal{L}$ be an extendable, factorial language.
For any $n \geq 1$ let $s(n) := p(n + 1) - p(n)$. For $v \in \mathcal{L}(n)$ let

$$m_l(v) = card\{a \in \Sigma, va \in \mathcal{L}(n + 1)\},$$
$$m_r(v) = card\{b \in \Sigma, bv \in \mathcal{L}(n + 1)\},$$
$$m_b(v) = card\{a \in \Sigma, b \in \Sigma, bva \in \mathcal{L}(n + 2)\}.$$

A word is call right special if $m_r(v) \geq 2$, left special if $m_l(v) \geq 2$ and bispecial if it is right and left special. Let $\mathcal{BL}(n)$ be the set of the bispecial words. Cassaigne [6] has shown:

Lemma 8.
$$s(n + 1) - s(n) = \sum_{v\in\mathcal{BL}(n)} m_b(v) - m_r(v) - m_l(v) + 1.$$

For the proof of the lemma we refer to [6] or [7].

Definition 9. *Consider the billiard map T inside the cube, and a point $(m, \omega) \in X_\infty$. We define the complexity $p(n, m, \omega)$ by the complexity of the infinite word $v_{m,\omega}$. We denote it by directional complexity.*

Definition 10. *A direction $\omega \in \mathbb{PR}^2$ is called a minimal direction if for all point m the sequence $(T^n(m, \omega) \cap \partial P)_{n\in\mathbb{N}}$ is dense in ∂P.*

Corollary 11. *For a minimal direction the complexity is independent of the initial point m.*

For the proof we refer to [12].
Notations: This corollary implies that we can omit the initial point in the notation $p(n, m, \omega)$, with the assumption that m is on a set where the orbit of (m, ω) is dense.

2.3. **Background and notations.** The following Lemma recall some usual results, see [8, 12].

Lemma 12. *Let* $\theta = \begin{pmatrix} a \\ b \end{pmatrix}$ *be an unit vector of* $\mathbb{R}^2$. *Consider a square coded with two letters, and the billiard map in this polygon.*

- *The direction* θ *is a minimal direction if and only if* a, b *are rationally independent over* $\mathbb{Q}$.
- *If the direction is not a minimal one, for all point* m *the billiard orbit of* (m, θ) *is periodic.*
- *If* θ *is a minimal direction then we obtain* $p(n, m, \theta) = n + 1$ *for all* m.
- *The orthogonal projection of a cubic billiard trajectory on a face of the cube is a billiard trajectory inside a square.*

Lemma 13. *In the cube a direction* ω *is a minimal direction if and only if the numbers* $(\omega_i)_{i \leq 3}$ *are independent over* $\mathbb{Q}$.

Definition 14. *An edge parallel to the axis* Ox, *respectively* Oy, *resp* Oz *is called of type 1, resp 2, resp 3.*

3. Proof of the Theorem

3.1. **First case.** We study the orbit of the point $m_0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ under T_ω. By unfolding we must compute the intersections of the line $m_0 + \mathbb{R}\omega$ with the three sort of faces. The calculus is similar in any case, thus we treat only one case: the intersection with the face $Y = k$ (same thing for the faces $X = l$ or $Z = m$ with $m, l, k \in \mathbb{Z}$). There exists λ such that $\begin{pmatrix} x + \lambda \\ y + \lambda\alpha \\ z + \lambda\beta \end{pmatrix}$ belongs to the face $Y = k$. We obtain $\lambda = \frac{k-y}{\alpha}$, we deduce that the intersection point is $\begin{pmatrix} x + \frac{k-y}{\alpha} \\ k \\ z + \frac{k-y}{\alpha}\beta \end{pmatrix}$. The point of the cube which corresponds in the unfolding to this point is $\begin{pmatrix} x + \frac{k-y}{\alpha} & mod 1 \\ 0 & \\ z + \frac{k-y}{\alpha}\beta & mod 1 \end{pmatrix}$.

To obtain the sequence of the orbit of (x, y, z) by T_ω, it remains to change $k \in \mathbb{Z}$. Since α, β are two rational numbers, we deduce that the sequence is periodic. Thus the trajectory is periodic, and the complexity is an eventually constant function.

3.2. **Case number 2.** Consider the projection on the plane Oxz. Since β is a rational number, we have a periodic trajectory in the square, see Lemma 12. Denote by (a_i) the periodic sequence of points inside the square, such that $a_p = a_1$, denote b_i the points of the cube such that $(a_i b_i)$ is parallel to the axis Oz. Consider the union S of the intervals

$$[a_i, a_{i+1}], [b_i, b_{i+1}], [a_i, b_i] \quad i \leq p - 1.$$

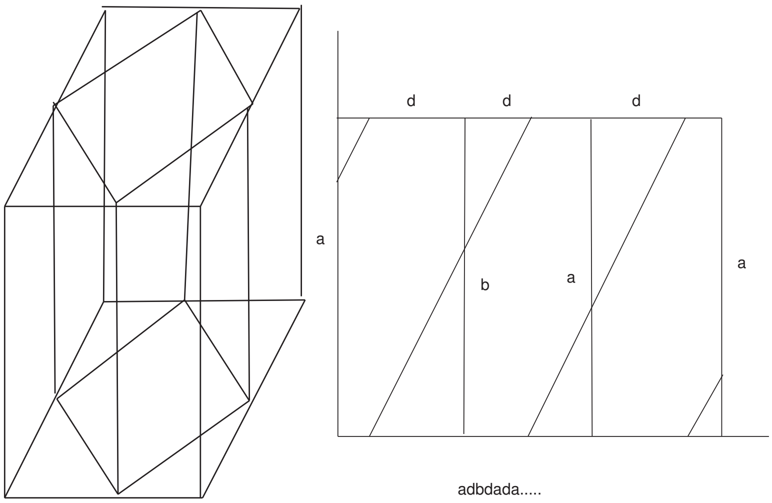

FIGURE 2. Billiard trajectory

The trajectory of (m_0, ω) is included in S, as can be seen by projection, see Figure 2. Now we unfold the trajectory. The unfolding of S is a rectangle. This rectangle is partitioned in several rectangles of the same shapes. The trajectory is a translation in this rectangle, see Figure 2. This translation is coded with three letters, see Figure 2, and it is minimal by hypothesis on α. If the translation was coded by two letters we would obtain a sturmian word. The computation of the complexity is reduced to the computation of the complexity of a translation: it is clearly sub linear. Moreover we remark that the rectangle S only depends on β by construction.

3.3. **Case number 3.** Consider the relation $a\alpha+b\beta+c=0$ with $a, b, c \in \mathbb{Z}$. We will study the orbit of $m_0 = (x, y, z)$ under T_ω. A point on this line has for coordinates $\begin{pmatrix} \lambda + x \\ \lambda\alpha + y \\ \lambda\beta + z \end{pmatrix}$. Thus it is in the plane $cX + aY + bZ = cx + ay + bz$. This plane intersect each united cube of the lattice $\mathbb{Z}^3$ on a polygon. By a translation each polygon is moved to the initial cube. This union of polygons contains the orbit of a point see Figure 3.

Lemma 15. *There are a finite number of such polygons inside the cube.*

Proof. We consider the intersection P of the plane with the initial cube. The other polygons are obtained by translating the intersection of P with another cube. Thus the study of the edges of the polygons in the face $Z = 0$ can be made by looking at the edges in the face $Z = k$, when k moves inside $\mathbb{Z}$. Consider the intersection of P with the face $Z = k$. We obtain a line of equation

$$\begin{cases} Z = k \\ cX + aY = cx + ay + bz - bk \end{cases}$$

The slope of this line is $\frac{-c}{a}$. It is a rational number. When k changes this slope is constant, thus all the edges in this face are parallel. Moreover the

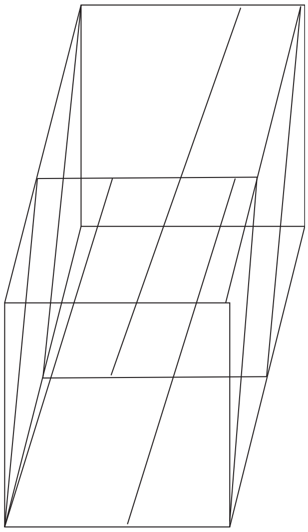

FIGURE 3. Billiard map inside the union of polygons

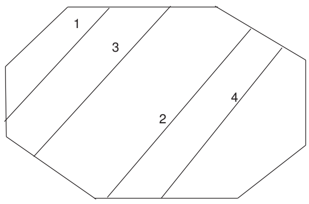

FIGURE 4. Linear flow inside a polygon

intersections of this line with the edges of the cube are obtained by replacing Y or X by an integer l. For example we obtain

$$X_{k,l} = \frac{cx + a + bz - bk - al}{c} = \frac{cx + ay + bz}{c} - \frac{bk + al}{c} \quad mod 1.$$

The set of all points is obtained by taking the union of k, l in $\mathbb{Z}$. This give a finite number of points since these numbers are rational. Thus in each face there are a finite number of parallel edges. Moreover inside two parallel faces the edges are parallel. □

Now the orbit of a point is included inside this finite union of polygons. The opposite sides of these polygons are parallel. Thus the billiard flow becomes a linear flow inside a polygon with parallel opposites sides see Figure 4. Thus we can apply the following result of Hubert.

Lemma 16. [9] *A minimal linear flow on a polygon with parallel opposite sides is of sub-linear complexity. Moreover the complexity does not depend on the initial point and the direction.*

Here we remark that several edges can be coded by the same letter, thus the complexity can be less than the initial one. Remark for finish the proof that the complexity only depends on the polygon. Thus it only depends of a, b, c.

3.4. Background of billiard complexity.

Definition 17. *In a polyhedron a generalized diagonal of direction ω between two edges is the union of all the billiards trajectory of direction ω between two points of these edges. We say it is of length n if each billiard trajectory hits n faces between the two points.*

If we fix the initial edge, we can describe the edges at length n, by the following result:

Lemma 18. [5] *Fix an edge A of the initial cube. The edge B is at length n of the edge A if and only if for all point $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ of B we have*

$$E(b_1) + E(b_2) + E(b_3) = n.$$

We recall the result of [4] which will be useful in the following.

Proposition 19. *Let ω be an unit vector, which is minimal for the cubic billiard, then we have for all integer n*

$$s(n+1,\omega) - s(n,\omega) = N(n,\omega),$$

where $N(n,\omega)$ is the number of generalized diagonals of direction ω and length n.

With the same hypothesis the next lemma proves that we can construct at most two diagonals of combinatorial length n in this direction.

Lemma 20. *If ω is minimal for the billiard map inside a cube, then we have*

$$N(n,\omega) \leq 2.$$

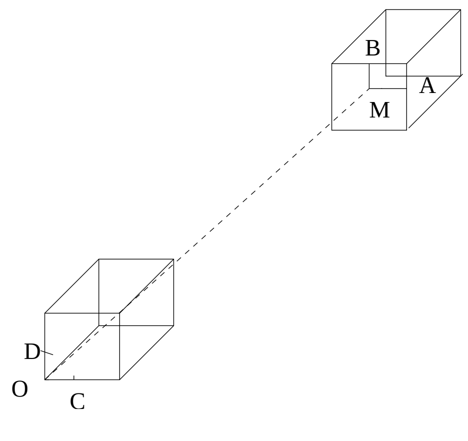

FIGURE 5. Diagonal

Let O be a vertex of the cube and consider the segment of direction ω who starts from O and ends at a point M after it passes through n cubes. M is a point of a face of an unfolding cube, if we translate M with a direction parallel to one of the two directions of the face we obtain a point A on an edge and if we call C the point such that $\vec{OC} = \vec{MA}$ then CA is a generalized diagonal, and we have another one, DB in the figure, arising from the second translation.

The symmetries of the cube implies that these diagonals are the only ones. It remains to prove that the two generalized diagonals are of combinatorial length n.

The first thing to remark is that the condition of total irrationality implies that a generalized diagonal can not begin and end on two parallel edges.

To see that the combinatorial length is at most n we can remark that the sum of the length of the projections is twice the length of the trajectory, so we just have to prove it for the projection, i.e. billiard in the square, where it follows from the symmetry.

3.5. **Case number 4.** In this section we will show that the number of generalized diagonals in the direction ω can be strictly less than two. First of all we recall the following lemma.

Lemma 21. *Consider three numbers a, b, c linearly independent over $\mathbb{Q}$. Assume that the following equation*

$$x/a + b/y + c/z = 0,$$

has an integer solution (x, y, z) with $x \neq 0$. Then the rational solutions of the equation are :

$$r(x', \frac{yx'}{x}, \frac{zx'}{x}) \quad x', r \in \mathbb{Q}.$$

Proof. Consider two solutions:

$$\begin{cases} x/a + y/b + z/c = 0 \\ x'/a + y'/b + z'/c = 0 \end{cases}$$

$$\begin{cases} x/a + y/b + z/c = 0 \\ (yx' - xy')/b + (zx' - xz')/c = 0 \end{cases}$$

Since b/c is an irrational number, we deduce

$$\begin{cases} x/a + y/b + z/c = 0 \\ yx' = xy' \\ zx' = xz' \end{cases}$$

$$\begin{cases} x/a + y/b + z/c = 0 \\ y' = yx'/x \\ z' = zx'/x \end{cases}$$

$$\begin{cases} y' = yx'/x \\ z' = zx'/x \end{cases}$$

□

Lemma 22. *Assume there exists n such that $N(n, \omega) < 2$, then :*

$$s(n + 1, \omega) - s(n, \omega) = 0.$$

Moreover there exists a line of direction ω which intersects the three types of edges and these three edges are in a fixed order, given by the direction.

Proof. First recall that the minimality of ω implies that the edges of a diagonal in direction ω can not be parallel. In all the rest of the proof we can assume that the edges of the generalized diagonal of direction ω are of type 1 and 3.
•Consider a trajectory in direction ω between two edges of types 1 and 3, consider the orthogonal reflection over the plane $X = Z$. This map exchanges the edges of type 1 and 3, but it remains invariant the edges of type 2. It implies that $N(n, \omega)$ is an even number, thus we can not have $N(n, \omega) = 1$. Thus we have $N(n, \omega) = 0$. Proposition 19 finishes the proof of the first part.

•By applying a translation we can always assume that the intersection points of the line $m + \mathbb{R}\omega$ with the edges of the cube have for coordinates

$$(x, 0, 0); (a, y, b); (c, d, z),$$

with x, y, z reals numbers and a, b, c, d integers. We obtain the system

$$\begin{cases} x + \lambda\omega_1 = a \\ \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ x + \mu\omega_1 = c \\ \mu\omega_2 = d \\ \mu\omega_3 = z \end{cases}$$

with λ, μ real numbers.

$$\begin{cases} x + \frac{b}{\omega_3}\omega_1 = a \\ \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ x + \frac{d}{\omega_2}\omega_1 = c \\ \mu\omega_2 = d \\ \mu\omega_3 = z \end{cases} \qquad \begin{cases} x = -\frac{b}{\omega_3}\omega_1 + a \\ \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ x = c - \frac{\omega_1}{\omega_2}d \\ \mu\omega_2 = d \\ \mu\omega_3 = z \end{cases}$$

$$\begin{cases} \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ \mu\omega_2 = d \\ \mu\omega_3 = z \\ x = a - b\frac{\omega_1}{\omega_3} \\ a - b\frac{\omega_1}{\omega_3} = c - d\frac{\omega_1}{\omega_2} \end{cases} \qquad \begin{cases} \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ \mu\omega_2 = d \\ \mu\omega_3 = z \\ x = a - b\frac{\omega_1}{\omega_3} \\ a - c = b\frac{\omega_1}{\omega_3} - d\frac{\omega_1}{\omega_2} \end{cases}$$

$$\begin{cases} \lambda\omega_2 = y \\ \lambda\omega_3 = b \\ \mu\omega_2 = d \\ \mu\omega_3 = z \\ x = a - b\frac{\omega_1}{\omega_3} \\ \frac{a-c}{\omega_1} = \frac{b}{\omega_3} - \frac{d}{\omega_2} \end{cases}$$

By hypothesis on ω, we have a relation of the form

$$\frac{A}{\omega_1}+\frac{B}{\omega_2}+\frac{C}{\omega_3}=0,$$

where $A,B,C\in\mathbb{Z}$ are relatively prime.

The last equation of the system is of the same form:

$$\frac{a-c}{\omega_1}+\frac{-b}{\omega_3}+\frac{d}{\omega_2}=0.$$

Thus we apply the preceding Lemma, and we deduce that A is a divisor of $a-$c and:

$$\begin{cases} d=B\frac{a-c}{A}\\ -b=C\frac{a-c}{A}\end{cases}$$

Finally the system becomes

$$\begin{cases} \lambda\omega_2=y\\ \lambda\omega_3=b\\ \mu\omega_2=d\\ \mu\omega_3=z\\ x=a-b\frac{\omega_1}{\omega_3}\\ d=B\frac{a-c}{A}\\ -b=C\frac{a-c}{A}\end{cases}$$

Thus this system has at least one solution, and the existence of the line is proved.

This system allow us to make several remarks. First the coordinates ω_i are positive numbers. This implies that A,B,C can not all be positive numbers. Assume that we have $A<0,B>0,C>0$ (the other cases are symmetric). We deduce that $a-c$ and d are of opposite sign, and that $a-c$ and b are of same sign.

• First assume $a-c\geq 0$ this implies

$$d\leq 0, b\geq 0.$$

From the system we deduce

$$\lambda\geq 0,\mu\leq 0.$$

This implies that the edges appear in the order $3;1;2$.

• The second case $a-c\leq 0$ implies by a similar argument that we have the order $2;1;3$.

Moreover the two orders are correlated, it depends the sense where we move along the line. Thus we can reduce to one order. □

Definition 23. *We label the three different faces of the cube by* $(v_i)_{i=1\dots3}$.

Corollary 24. *Assume* ω *is a minimal direction and fulfills*

$$\frac{A}{\omega_1}=\frac{B}{\omega_2}+\frac{C}{\omega_3}\quad A,B,C\in\mathbb{N}^*.$$

Then for all integer n we have the dichotomy: If the billiard's orbit of the origin, at the step n, meets a face labelled by v_1 and if A divides n, then $s(n+1,\omega)-s(n,\omega)=0$, either $s(n+1)-s(n)=2$.

Proof. First we claim that there exists an infinite number of integers n such that $s(n)=0$. Indeed in the last system obtained in the proof of Lemma 22 we can modify the values of a, c such that A divides $a-c$. Now we can assume that the order related to the edges is $3;1;2$ see Lemma 22. Consider the orbit of the origin, and the intersection with a face (of a cube of $\mathbb{Z}^3$) parallel to $X=0$. With the method of Lemma 20 we deduce that the only possibility for a diagonal is a trajectory between edges 3 and 2. Denote by n the length of the diagonal, by preceding system we deduce that if A divides n the trajectory between 3 and 2 pass through the edge 1. We deduce $s(n+1,\omega)=s(n,\omega)$. The first part is proved.

Assume now that we meet another face at step n, for example the face parallel to $Z=0$. Then the two associated diagonals have for order $1;2$ and $2;1$. We prove by contradiction that we can not have $N(n,\omega)\leq 1$. Since the order is unique, see Lemma 22, the only possibility to obtain a third edge is to start from the edge labelled 1. Then the diagonal which start form 3 does not intersect another edge. This implies $N(n,\omega)=1$, but this is a contradiction with the first part of Lemma 22. □

This corollary implies that the sequence $(s(n,\omega))_{n\in\mathbb{N}}$ can take only two values. Due to the next lemma, to finish the proof it remains to obtain the frequency of each value.

Lemma 25. *Assume that the sequence $(s(n+1,\omega)-s(n,\omega))_{n\in\mathbb{N}}$ has value in $\{0;1;2\}$, and that the numbers $0;1;2$ have respectively for frequency l,m,p. Then the complexity satisfy*

$$p(n)\sim\frac{m+2p}{2}n^2.$$

Lemma 26. *Assume the direction satisfy the hypothesis $\frac{A}{\omega_1}=\frac{B}{\omega_2}+\frac{C}{\omega_3}$, with $A,B,C\in\mathbb{N}$. Then the frequency of 0 in the sequence $(s(n+1,\omega)-s(n,\omega))_{n\in\mathbb{N}}$ is:*

$$f_0=\frac{\omega_1}{A(\omega_1+\omega_2+\omega_3)}.$$

Proof. By Corollary 24 it is equivalent to consider the intersection of the orbit of the origin with the planes parallel to $X=0$. A point in the orbit of the origin has for coordinates:

$$\begin{pmatrix}\lambda\omega_1\\ \lambda\omega_2\\ \lambda\omega_3\end{pmatrix}.$$

It meet the face $X=iA$ at the point

$$\begin{pmatrix}iA\\ \frac{iA}{\omega_1}\omega_2\\ \frac{iA}{\omega_1}\omega_3\end{pmatrix}.$$

Then we must compute the number of i such that this point is at combinatorial length less than n. By Lemma 18 it remains to compute

$$card\{i|iA+[\frac{iA}{\omega_1}\omega_2]+[\frac{iA}{\omega_1}\omega_3]\leq n\}.$$
$$=\frac{n\omega_1}{A(\omega_1+\omega_2+\omega_3)}+o(n).$$
$$f_0=\frac{\omega_1}{A(\omega_1+\omega_2+\omega_3)}.$$

□

The proof of the Theorem is a consequence of the two preceding Lemmas.

3.6. **Last case.** The proof can be found in [4] or in [3] for the s dimensional case. In the first article the computation is made by the proof that $N(n,\omega)=2$ for all integer.

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